Markov-Switching Models with Evolving Regime-Specific Parameters: Are Post-War Booms or Recessions All Alike?

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Abstract

In this paper, we relax the assumption of constant regime-specific mean growth rates in Hamilton's (1989) two-state Markov-switching model of the business cycle. We first present a benchmark model, in which each regime-specific mean growth rate evolves according to a random walk process over different episodes of booms or recessions. We then present a model with vector error correction dynamics for the regime-specific mean growth rates, by deriving and imposing a condition for the existence of a long-run equilibrium growth rate for real output. In the Bayesian Markov Chain Monte Carlo (MCMC) approach developed in this paper, the counterfactual priors, as well as the hierarchical priors for the regime-specific parameters, play critical roles.

By applying the proposed model and approach to the postwar real GDP growth data (1947Q4-2011Q3), we uncover the evolving nature of the regime-specific mean growth rates of real output in the U.S. business cycle. An additional feature of the postwar U.S. business cycle that we uncover is a steady decline in the long-run equilibrium output growth. The decline started in the mid-1950s and ended in the mid-1980s, coinciding with the beginning of the Great Moderation. Our empirical results also provide partial, if not decisive, evidence that the central bank may have been more successful in restoring the economy back to its long-run equilibrium growth path after unusually severe recessions than after unusually good booms.

Key Words: Bayesian Approach, Business Cycle, Counterfactual Prior, Evolving Regime-Specific Parameters, Hierarchical Prior, Markov Switching, Hamilton Model, MCMC, State-Space Model

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1. Introduction

Blanchard and Watson (1986) raised an interesting question of whether or not business cycles are all alike. Their answer was "No." To motivate this paper, we first ask, "Are post-war booms or recessions all alike?" Our answer is tentatively "No." In a two-state Markov-switching model of the business cycle as proposed by Hamilton (1989), the mean growth rates of real GDP during different episodes of a specific regime (boom or recession) are assumed to be the same. We claim that, even though this assumption may be a reasonable approximation for a specific sample, it may be a poor approximation for the extended sample that covers the whole postwar period. This is confirmed by Figure 1, in which the quarterly growth rates of real GDP for the sample period 1947Q4 to 2011Q3 are plotted along with the mean growth rate for each episode of NBER boom or recession. The shaded areas refer to the NBER recession periods. In the summary statistics provided in Table 1, the mean growth rates for the 12 historical episodes of booms range between 0.59 and 1.83 with a standard deviation of 0.37. The mean growth rates for the 11 historical episodes of recessions range between 0.02 and -0.69 with a standard deviation of 0.23.

In this paper, we propose a flexible two-state Markov-switching model of the business cycle, in which the regime-specific mean growth rates of real output may evolve over different episodes of booms or recessions. That is, we propose a new model of the business cycle that consists of three features: i) specification of the Markov-switching latent variable that determines the business cycle regimes; ii) specification of the evolving regime-specific parameters in the form of hierarchical priors; and iii) specification of the time series within each regime.

We first present a benchmark model, in which we assume a simple random walk hierarchical prior for each regime-specific mean growth rate. Within this framework, we provide insights into how the inferences about the model can be made. One potential difficulty is that, conditional on the current state being a recession (boom), the prior for the mean growth rate for a boom (recession) is not defined. We propose to solve the problem by employing 'counterfactual priors' that are appropriately derived from the hierarchical priors. For example, conditional on the current state being a boom, we ask what the mean growth

rate would be if we were in a recession.

By imposing a condition for the existence of a long-run or unconditional growth rate for real output, we then extend the benchmark model to allow for a cointegrating relationship between the two regime-specific mean growth rates. For this purpose, we design the hierarchical priors and the corresponding counterfactual priors in order to incorporate vector error correction dynamics for the regime-specific mean growth rates. Note that the long-run restriction incorporated in the extended model can result from the central bank's successful attempts to stabilize the economy. For example, if the economy deviates from the long-run growth path due to a large and infrequent shock, the central bank may intervene to restore the economy back to its long-run equilibrium growth path.

For inference of the models proposed, we build on recent advances in Bayesian approaches to change-point models that allow for flexible relationships between parameters in various regimes and/or unknown number of structural breaks. (Koop and Potter (2007), Giordani and Kohn (2008), Geweke and Jiang (2009), etc.) In particular, we follow Koop and Potter (2007) and cast the models into standard Markov-switching state-space formulations with heteroscedastic shocks to regime specific parameters. The counterfactual priors, as well as the hierarchical priors, play important roles in this step. Once the models are put into standard state-space formulations, a Markov Chain Monte Carlo (MCMC) procedure can be easily developed based on the existing posterior simulation method for state-space models and that for Markov-switching models. For example, in order to generate the evolving regime-specific parameters conditional on the Markov-switching regime indicator variable, we can take advantage of Carter and Kohn's (2007) and Kim et al.'s (1998) methods of posterior simulation for linear state-space models. In order to generate the Markov-switching regime indicator variable conditional on the evolving regime-specific parameters, we employ a modified version of Albert and Chib's (1993) method.

The remainder of this paper is organized as follows. In Section 2, we briefly review recent advances in the Bayesian approach to change-point models. Section 3 presents model specifications. We first present a benchmark Markov-switching model, in which the regime-specific parameters are assumed to follow random walks over different episodes of regimes. We then extend the benchmark model to a general case, in which the regime-specific parameters are

assumed to be cointegrated. In this case, the hierarchical priors for the regime-specific parameters, combined with the counterfactual priors, form a vector error correction model. In Section 4, we present a state-space representation of the general model, and develop the MCMC procedure for Bayesian inference of the model. In Section 5, we apply the model to postwar U.S. real GDP growth. Section 6 provides a summary.

2. Hierarchical Priors in Bayesian Approaches to Change-Point Models: Review

In order to provide some econometric foundation for the current paper, we begin our discussion by considering the following simplified version of a change-point model with $M-1$ structural breaks or M regimes:

$$y_t = \mu_{D_t} + x_t, \qquad D_t = 1, 2, ..., M, \tag{1}$$

$$\phi(L)x_t = e_t, \quad e_t \sim i.i.d.N(0, \sigma_e^2), \tag{2}$$

where all roots of $\phi(L) = 1 - \phi_1 L - \ldots - \phi_r L^r = 0$ lie outside the complex unit circle; D_t specifies the regimes separated by the change points. By assuming that the latent variable D_t is Markovian with absorbing states, Chib (1998) deals with the case of a fixed (known) number of regimes M and independent parameters across regimes. Pesaran et al. (2006) assume that all the $\mu_{D_t}'s$ are drawn from a common distribution. More recently, Koop and Potter (2007) extend Chib's (1998) model in at least two directions. First, they consider the case of an unknown number of structural breaks or regimes by employing a flexible Poisson hierarchical prior distribution for the durations of the regimes. Second, for given M and conditional on $D_t = \tau$, they allow for dependence between the pre-and post-break parameters of the model by employing a hierarchical prior of the following form:

$$\mu_\tau = \mu_{\tau-1} + \omega_\tau, \quad \omega_\tau \sim i.i.d.N(0, \Sigma_\omega), \quad \tau = 1, 2, ..., M \tag{3}$$

The strategy adopted by Koop and Potter (2007) is to put the equations in (1)-(3) into a standard state-space model used in the unobserved-components or time-varying parameters formulations. Then, conditional on the dates of structural breaks, the methods of posterior

simulation for state-space models are readily available, as developed by Carter and Kohn (1994) and Kim et al. (1998).

Note that the model in equations (1)-(3) is different from the standard state-space model in that the regime-specific parameters in equation (3) do not have the t subscripts. Conditional on the dates of structural breaks, the standard state-space representation of the model in equations (1)-(3) is given below:

$$y_t = \mu_t^* + x_t, \tag{4}$$

$$\mu_t^* = \mu_{t-1}^* + \omega_t^*, \quad \omega_t^* \sim N(0, d_t\Sigma_\omega), \tag{5}$$

where x_t is as defined in (2) and

$$d_t = \begin{cases} 1, & \text{if } \ D_{t-1} = i \ \text{ and } \ D_t = j \text{ with } j = i+1 \ ; \\ 0, & \text{if } D_{t-1} = i \ \text{ and } \ D_t = j \text{ with } j = i, \end{cases} \tag{6}$$

which suggests that μ_t^* is subject to a heteroscedastic shock. μ_t^* changes only when regime-shift occurs and is constant otherwise.

In the next section, we adopt the above framework in specifying and making inferences of the Markov-switching models with evolving regime-specific parameters. The proposed model can be thought of as an extension of Koop and Potter (2007) to the case of a Markov-switching model. According to their terminology, the mean growth rate for recession or boom undergoes a structural break whenever we face a new episode of recession or boom.

3. Markov-Switching Models with Evolving Regime-Specific Parameters

3.1. A Benchmark Model with Random Walk Dynamics for Regime-Specific Parameters

Let y_t be real output growth, and consider the following Markov-switching model of the business cycle:

$$y_t = (1 - S_t)\mu_{0,\tau_0} + S_t\mu_{1,\tau_1} + x_t, \quad S_t = 0, 1, \tag{7}$$

$$\phi(L)x_t = e_t, \quad e_t \sim i.i.d.N(0, \sigma_e^2), \tag{8}$$

$$t = 1, 2, ..., T; \quad \tau_0 = 1, 2, ..., N_0; \quad \tau_1 = 1, 2, ..., N_1,$$

where μ_{0,τ_0} is the mean growth rate during the $\tau_0 - th$ episode of boom in the sample; μ_{1,τ_1} is the mean growth rate during the $\tau_1 - th$ episode of recession; N_0 and N_1 are the total numbers of the episodes of booms and recessions, respectively, conditional on the states; and the roots of $\phi(L) = 1 - \phi_1 L - \ldots - \phi_r L^r = 0$ lie outside the complex unit circle. Note that N_0 and N_1 are random variables, and they are dependent upon the realizations of the latent state variables $\tilde{S}_T = [\, S_1 \quad S_2 \quad \ldots \quad S_T \,]'$ that characterize the business cycle regime. The latent state variable S_t follows a first-order Markov-switching process with the transition probabilities:

$$Pr[S_t = 1 | S_{t-1} = 1] = p, \quad Pr[S_t = 0 | S_{t-1} = 0] = q. \tag{9}$$

While Hamilton (1989) assumes that $\mu_{0,\tau_0} = \mu_0$ for all $\tau_0 = 1, 2, ..., N_0$ and $\mu_{1,\tau_1} = \mu_1$ for all $\tau_1 = 1, 2, ..., N_1$, we allow for the possibility that different episodes of booms (or recessions) have different mean growth rates. In order to allow for dependence of mean growth rates between current and past episodes of booms or recessions, we adopt hierarchical priors given by the following random walk dynamics for μ_{0,τ_0} and μ_{1,τ_1}:

Hierarchical Priors

$$\mu_{0,\tau_0} = \mu_{0,\tau_0 - 1} + \omega_{0,\tau_0}, \quad \omega_{0,\tau_0} \sim i.i.d.N(0, \sigma_{\omega,0}^2), \tag{10}$$

$$\mu_{1,\tau_1} = \mu_{1,\tau_1 - 1} + \omega_{1,\tau_1}, \quad \omega_{1,\tau_1} \sim i.i.d.N(0, \sigma_{\omega,1}^2), \tag{11}$$

$$\tau_0 = 1, 2, ..., N_0; \qquad \tau_1 = 1, 2, ..., N_1,$$

where ω_{0,τ_0} and ω_{1,τ_1} are independent of each other and are not correlated with e_t in equation (8). Within the context of the linear models with multiple structural breaks, Koop and Potter (2007) employ the same hierarchical prior in order to allow for dependence in parameters across regimes. When $\sigma_{\omega,0}^2 = \sigma_{\omega,1}^2 = 0$ the above model collapses to that of Hamilton (1989). The fundamental difference between the model proposed in this paper and that in Hamilton (1989) is illustrated in Figure 2.

The model in equations (7)-(8) and (10)-(11) differs from a standard state-space model in that the subscripts on the parameters of the measurement equation in (7) do not have t subscripts but rather τ_0 and τ_1 subscripts, so that the regime-specific parameters $\mu_{0,\tau 0}$ or μ_{1,τ_1} change only when we face a new episode of boom or recession. Thus, in adopting Koop and Potter's (2007) approach, successful inference of the model would depend upon a successful derivation of its conventional unobserved-components representation of the following form:

<u>*Conventional Unobserved-Components Model Representation*</u>

$$y_t = (1 - S_t)\mu_{0,t}^* + S_t\mu_{1,t}^* + x_t \tag{12}$$

where the dynamics of μ_{0,τ_0} in equation (10) should be captured by $\mu_{0,t}^*$ and the dynamics of μ_{1,τ_1} in equation (11) should be captured by $\mu_{1,t}^*$. Note that in the above formulation, all the variables have t subscripts.

However, μ_{0,τ_0} is defined only during booms and not during recessions, resulting in difficulty in deriving the dynamics of $\mu_{0,t}^*$ during recessions. In the same way, μ_{1,τ_1} is defined only during recessions and not during booms, resulting in difficulty in deriving the dynamics of $\mu_{1,t}^*$ during booms. In order to overcome this difficulty, we employ the concept of 'counterfactual priors', by asking: i) Conditional on the current state being the $\tau_0 - th$ boom, what would be the mean growth of real GDP if we were in a recession? (μ_{1,τ_0}); and ii) Conditional on the current state being the $\tau_1 - th$ recession, what would be the mean growth of real GDP if we were in a boom? (μ_{0,τ_1}). These counterfactual priors, as implied by the random-walk hierarchical priors in (10) and (11) are given by:

<u>*Counterfactual Priors*</u>

$$\mu_{1,\tau_0} = \mu_{1,\tau_1'}, \quad \tau_0 = 1, 2, ..., N_0, \tag{13}$$

$$\mu_{0,\tau_1} = \mu_{0,\tau_0'}, \quad \tau_1 = 1, 2, ..., N_1, \tag{14}$$

where $\mu_{1,\tau_1'}$ is the mean growth rate during a recession right before the $\tau_0 - th$ episode of boom and $\mu_{0,\tau_0'}$ is the mean growth rate during a boom right before the $\tau_1 - th$ episode of recession.

As illustrated in Figure 3, the hierarchical priors in equations (10)-(11) and the resulting counterfactual priors in equations (13)-(14) can be combined together. Thus the model given by equations (7), (10)-(11), and (13)-(14) can be rewritten as:

$$y_t = (1 - S_t)\mu_{0,\tau} + S_t\mu_{1,\tau} + x_t, \quad S_t = 0, 1, \tag{7'}$$

$$\mu_{0,\tau} = \mu_{0,\tau-1} + \omega_{0,\tau}, \quad \omega_{0,\tau} \sim N(0, (1 - S_t)\sigma^2_{\omega,0}), \tag{15}$$

$$\mu_{1,\tau} = \mu_{1,\tau-1} + \omega_{1,\tau}, \quad \omega_{1,\tau} \sim N(0, S_t\sigma^2_{\omega,1}), \tag{16}$$

$$\tau = 1, 2, ..., N_0 + N_1, \quad t = 1, 2, ..., T,$$

where, conditional on the current state being a boom ($S_t = 0$), we have $\mu_{0,\tau} = \mu_{0,\tau_0}$ (prior); $\mu_{1,\tau} = \mu_{1,\tau_0}$ (counterfactual prior); $\mu_{0,\tau-1} = \mu_{0,\tau_1'}$; $\mu_{1,\tau-1} = \mu_{1,\tau_1'}$; $\omega_{0,\tau} = \omega_{0,\tau_0}$; and $\omega_{1,\tau} = 0$. Conditional on the current state being a recession ($S_t = 1$), we have $\mu_{0,\tau} = \mu_{0,\tau_1}$ (counterfactual prior); $\mu_{1,\tau} = \mu_{1,\tau_1}$ (prior); $\mu_{0,\tau-1} = \mu_{0,\tau_0'}$; $\mu_{1,\tau-1} = \mu_{1,\tau_0'}$; $\omega_{0,\tau} = 0$; and $\omega_{1,\tau} = \omega_{1,\tau_1}$. Furthermore, note that equations (15)-(16) imply the following random walk dynamics with heteroscedastic disturbances for $\mu^*_{0,t}$ and $\mu^*_{1,t}$ in equation (12):

$$\mu^*_{0,t} = \mu^*_{0,t-1} + \omega^*_{0,t}, \quad \omega^*_{0,t} \sim N(0, d_{10,t}\sigma^2_{\omega,0}), \tag{17}$$

$$\mu^*_{1,t} = \mu^*_{1,t-1} + \omega^*_{1,t}, \quad \omega^*_{1,t} \sim N(0, d_{01,t}\sigma^2_{\omega,1}), \tag{18}$$

$$t = 1, 2, ..., T,$$

where

$$d_{ij,t} = \begin{cases} 1, & \text{if } \; S_{t-1} = i \textit{ and } S_t = j, \quad j \neq i; \\ 0, & \text{otherwise,} \end{cases} \tag{19}$$

and for identification of the model, we need

$$\mu^*_{0,t} > \mu^*_{1,t}, \quad \forall \;\; t. \tag{20}$$

3.2. An Extended Model with a Long-Run Restriction: Vector Error Correction Dynamics for Mean Growth Rates

One potential weakness of our benchmark model in Section 3.1 is that the long-run or the unconditional expectation of the output growth rate does not exist. In this section, we first derive a condition for the existence of a long-run growth rate.

By denoting the long-run growth rate as δ, we rewrite equation (7) as

$$y_t = \delta + (1 - S_t)\mu_{0,\tau_0} + S_t\mu_{1,\tau_1} + x_t. \tag{21}$$

Assume that, at time t, we are under $\tau_j - th$ episode of boom ($j = 0$) or recession ($j = 1$). Given the random walk hierarchical priors and the counterfactual priors implied by them as in Section 3.1, we have:

$$E(\mu_{0,\tau_{S_{t+1}}}|I_{\tau_j}) = \mu_{0,\tau_j}, \quad j = 0, 1 \tag{22}$$

$$E(\mu_{1,\tau_{S_{t+1}}}|I_{\tau_j}) = \mu_{1,\tau_j}, \quad j = 0, 1 \tag{23}$$

where I_{τ_j} refers to all the past and current regime-specific mean growth rates up to current episode of boom or recession. These results lead to the following prediction of the mean growth rate at time $t + 1$:

$$\begin{aligned} &E(y_{t+1}|I_{\tau_j}) \\ &\quad = \delta + (1 - E(S_{t+1}|I_{\tau_j}))E(\mu_{0,\tau_{S_{t+1}}}|I_{\tau_j}) + E(S_{t+1}|I_{\tau_j})E(\mu_{1,\tau_{S_{t+1}}}|I_{\tau_j}) + E(x_t|I_{\tau_j}) \\ &\quad = \delta + Pr[S_{t+1} = 0|I_{\tau_j}]\mu_{0,\tau_j} + Pr[S_{t+1} = 1|I_{\tau_j}]\mu_{1,\tau_j} + E(x_t|I_{\tau_j}), \qquad j = 0, 1 \end{aligned} \tag{24}$$

By taking unconditional expectations on both sides of equation (24), we get the following restriction for the existence of the unconditional expectation of the growth rate:

$$E(\pi_0\mu_{0,\tau} + \pi_1\mu_{1,\tau}) = 0, \tag{25}$$

where, conditional on $S_t = 0$, we have $\mu_{0,\tau} = \mu_{0,\tau_0}$ (prior) and $\mu_{1,\tau} = \mu_{1,\tau_0}$ (counterfactual prior); conditional on $S_t = 1$, we have $\mu_{0,\tau} = \mu_{0,\tau_1}$ (counterfactual prior) and $\mu_{1,\tau} = \mu_{1,\tau_1}$ (prior); and $\pi_i = Pr[S_{t+1} = i]$, $i = 0, 1$, are the unconditional probabilities of boom ($i = 0$) and recession ($i = 1$). Notice that this long-run restriction, combined with the random

walk assumptions for the regime-specific mean growth rates, suggests that $\tau_{0,\tau}$ and $\tau_{1,\tau}$ are cointegrated with a cointegrating vector $[\,\pi_0 \quad \pi_1\,]'$.

In this section, we impose the above long-run restriction in the benchmark model, by considering the following vector error correction dynamics for the regime-specific mean growth rates:

<u>*Hierarchical Priors*</u>

$$\mu_{0,\tau_0} = \mu_{0,\tau_1'} + \theta_0(\pi_0\mu_{0,\tau_1'} + \pi_1\mu_{1,\tau_1'}) + \omega_{0,\tau_0}, \quad \omega_{0,\tau_0} \sim i.i.d.N(0, \sigma^2_{\omega,0}), \tag{26}$$

$$\mu_{1,\tau_1} = \mu_{1,\tau_0'} + \theta_1(\pi_0\mu_{0,\tau_0'} + \pi_1\mu_{1,\tau_0'}) + \omega_{1,\tau_1}, \quad \omega_{1,\tau_1} \sim i.i.d.N(0, \sigma^2_{\omega,1}), \tag{27}$$

$$\tau_0 = 1, 2, ..., N_0; \quad \tau_1 = 1, 2, ..., N_1$$

where $\mu_{1,\tau_1'}$ is the mean growth rate during a recession right before the $\tau_0 - th$ episode of boom and $\mu_{0,\tau_1'}$ is the counterfactual mean growth rate of a boom during the same recession period; $\mu_{0,\tau_0'}$ is the mean growth rate during a boom right before the $\tau_1 - th$ episode of recession and $\mu_{1,\tau_0'}$ is the counterfactual mean growth rate of a recession during the same boom period.

It is straightforward to derive the dynamics for the counterfactual priors as implied by the above hierarchical priors. They are given below:

<u>*Counterfactual Priors*</u>

$$\mu_{1,\tau_0} = \mu_{1,\tau_1'} + \theta_1(\pi_0\mu_{0,\tau_1'} + \pi_1\mu_{1,\tau_1'}), \quad \tau_0 = 1, 2, ..., N_0, \tag{28}$$

$$\mu_{0,\tau_1} = \mu_{0,\tau_0'} + \theta_0(\pi_0\mu_{0,\tau_0'} + \pi_1\mu_{1,\tau_0'}), \quad \tau_1 = 1, 2, ..., N_1. \tag{29}$$

Note that, when $\theta_0 = \theta_1 = 0$, the hierarchical priors and the counterfactual priors specified in equations (26)-(29) collapse to those in equations (10)-(11) and (13)-(14).

What follows briefly describes the nature of the model with the long-run restriction. Suppose that, during the last boom, the economy was operating at the long-run equilibrium in the sense that $\pi_0\mu_{0,\tau_0-1} + \pi_1\mu_{1,\tau_0-1} = 0$. Further suppose that the following recession was unusually severe in the sense that $\pi_0\mu_{0,\tau_1'} + \pi_1\mu_{1,\tau_1'} < 0$. Then, the central bank may

intervene to restore the economy back to long-run equilibrium growth path, resulting in a higher growth during the $\tau_0 - th$ boom than otherwise. In this case, we can predict $\theta_0 < 0$. In the same spirit, if the central bank responds to an unusually high growth rate during a boom (preceding the current recession) in the opposite way, we can also predict $\theta_1 < 0$.

By combining the hierarchical priors in equations (26)-(27) and the counterfactual priors in (28)-(29), we can rewrite the model given by equations (21) and (26)-(29) as:

$$y_t = \delta + (1 - S_t)\mu_{0,\tau} + S_t\mu_{1,\tau} + x_t, \tag{21$'$}$$

$$\mu_{0,\tau} = \mu_{0,\tau-1} + \theta_0(\pi_0\mu_{0,\tau-1} + \pi_1\mu_{1,\tau-1}) + \omega_{0,\tau}, \quad \omega_{0,\tau} \sim N(0, (1 - S_t)\sigma^2_{\omega,0}), \tag{30}$$

$$\mu_{1,\tau} = \mu_{1,\tau-1} + \theta_1(\pi_0\mu_{0,\tau-1} + \pi_1\mu_{1,\tau-1}) + \omega_{1,\tau}, \quad \omega_{1,\tau} \sim N(0, S_t\sigma^2_{\omega,1}), \tag{31}$$

$$\mu_{0,\tau} > 0 \;\; and \;\; \mu_{1,\tau} < 0, \;\; \forall \;\; \tau,$$

$$\tau = 1, 2, ..., N_0 + N_1, \quad t = 1, 2, ..., T,$$

where, conditional on the current state being a boom ($S_t = 0$), we have: $\mu_{0,\tau} = \mu_{0,\tau_0}$ (prior); $\mu_{1,\tau} = \mu_{1,\tau_0}$ (counterfactual prior); $\mu_{0,\tau-1} = \mu_{0,\tau_1'}$; $\mu_{1,\tau-1} = \mu_{1,\tau_1'}$; $\omega_{0,\tau} = \omega_{0,\tau_0}$; and $\omega_{1,\tau} = 0$. Conditional on the current state being a recession ($S_t = 1$), we have: $\mu_{0,\tau} = \mu_{0,\tau_1}$ (counterfactual prior); $\mu_{1,\tau} = \mu_{1,\tau_1}$ (prior); $\mu_{0,\tau-1} = \mu_{0,\tau_0'}$; $\mu_{1,\tau-1} = \mu_{1,\tau_0'}$; $\omega_{0,\tau} = 0$; and $\omega_{1,\tau} = \omega_{1,\tau_1}$. Then, as in the previous section and as illustrated in Figure 4, by noting that (30)-(31) imply vector error correction dynamics with heteroscedastic shocks, we have the following conventional unobserved-components representation of the model:

Conventional Unobserved-Components Model Representation

$$y_t = \delta + (1 - S_t)\mu^*_{0,t} + S_t\mu^*_{1,t} + x_t, \tag{32}$$

$$\mu^*_{0,t} = \mu^*_{0,t-1} + \theta_0(d_{10,t} + d_{01,t})(\pi_0\mu^*_{0,t-1} + \pi_1\mu^*_{1,t-1}) + \omega^*_{0,t}, \quad \omega^*_{0,t} \sim N(0, d_{10,t}\sigma^2_{\omega,0}), \tag{33}$$

$$\mu^*_{1,t} = \mu^*_{1,t-1} + \theta_1(d_{10,t} + d_{01,t})(\pi_0\mu^*_{0,t-1} + \pi_1\mu^*_{1,t-1}) + \omega^*_{1,t}, \quad \omega^*_{1,t} \sim N(0, d_{01,t}\sigma^2_{\omega,1}), \tag{34}$$

$$t = 1, 2, ..., T,$$

where $d_{ij,t}$ is as defined in equation (19), and for identification of the model, we need

$$\mu_{0,t}^* > 0 \;\; and \;\; \mu_{1,t}^* < 0, \quad \forall \;\; t. \tag{35}$$

Finally, in order to guarantee the stability of the above vector error correction model and the existence of long-run output growth, we actually need a restriction on the θ_0 and θ_1 parameters. If we cast the vector error-correction model in (30)-(31) into state-space form, we have:

$$\begin{bmatrix} \Delta\mu_{0,\tau} \\ \Delta\mu_{1,\tau} \\ z_\tau \end{bmatrix} = \begin{bmatrix} 0 & 0 & \theta_0 \\ 0 & 0 & \theta_1 \\ 0 & 0 & 1+\theta_0\pi_0+\theta_1\pi_1 \end{bmatrix} \begin{bmatrix} \Delta\mu_{0,\tau-1} \\ \Delta\mu_{1,\tau-1} \\ z_{\tau-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \pi_0 & \pi_1 \end{bmatrix} \begin{bmatrix} \omega_{0,\tau} \\ \omega_{1,\tau} \end{bmatrix}, \tag{36}$$

$$\begin{bmatrix} \omega_{0,\tau} \\ \omega_{1,\tau} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} (1-S_t)\sigma^2_{\omega,0} & 0 \\ 0 & S_t\sigma^2_{\omega,1} \end{bmatrix} \right), \tag{37}$$

$$\tau = 1, 2, ..., N_0 + N_1,$$

where $z_\tau = \pi_0\mu_{0,\tau} + \pi_1\mu_{1,\tau}$ is the equilibrium error during period τ. As the equilibrium error needs to be stationary, the restriction on the θ_0 and θ_1 parameters are given by:

$$-1 < 1 + \theta_0\pi_0 + \theta_1\pi_1 < 1 \tag{38}$$

4. A Markov-Chain Monte Carlo (MCMC) Procedure

4.1. Outline for the MCMC Procedure

As in Koop and Potter (2007), we first cast the unobserved components model derived in the previous section into a state-space model. For illustrative purposes, we assume that x_t in equation (21) or (32) follows a white noise process with $\phi(L) = 1$.

Measurement Equation

$$y_t = \delta + [\,(1 - S_t) \quad S_t\,] \begin{bmatrix} \mu^*_{0,t} \\ \mu^*_{1,t} \end{bmatrix} + e_t, \quad e_t \sim i.i.d.N(0, \sigma_e^2), \tag{39}$$

$$\left(\Leftrightarrow \quad y_t = \delta + H_t \mu^*_t + e_t, \quad e_t \sim i.i.d.N(0, \sigma_e^2)\right)$$

<u>*State Equation*</u>

$$\begin{bmatrix} \mu^*_{0,t} \\ \mu^*_{1,t} \end{bmatrix} = \begin{bmatrix} 1 + \theta_0 \pi_0 (d_{10,t} + d_{01,t}) & \theta_0 \pi_1 (d_{10,t} + d_{01,t}) \\ \theta_1 \pi_0 (d_{10,t} + d_{01,t}) & 1 + \theta_1 \pi_1 (d_{10,t} + d_{01,t}) \end{bmatrix} \begin{bmatrix} \mu^*_{0,t-1} \\ \mu^*_{1,t-1} \end{bmatrix} + \begin{bmatrix} \omega^*_{0,t} \\ \omega^*_{1,t} \end{bmatrix} \tag{40}$$

$$\left(\Leftrightarrow \quad \mu^*_t = F_t \mu^*_{t-1} + \omega_t, \quad \omega_t \sim N(0, \Omega_t)\right),$$

where $\Omega_t = Diag\,(\,d_{10,t}\sigma^2_{\omega,0},\ d_{01,t}\sigma^2_{\omega,1}\,)$ and $d_{ij,t}$ is as defined in equation (19).

Conditional on $\tilde{S}_T = [\,S_1 \quad S_2 \quad \ldots \quad S_T\,]'$, the above is a linear state-space model with heteroscedastic shocks, and a procedure for making inferences on $\mu^*_{0,t}$ and $\mu^*_{1,t}$ (the elements of the state vector μ^*_t) can easily be developed by modifying the procedure proposed by Carter and Kohn (1994). Furthermore, conditional on the $\mu^*_{0,t}$ and $\mu^*_{1,t}$ terms generated for $t = 1, 2, ..., T$, a procedure for generating the regime indicator variable S_t can be easily derived by modifying the procedure proposed by Albert and Chib (1993). In what follows, we provide a summary of the prior employed for Bayesian inference of the model and present an outline for the MCMC procedure.

By defining $\tilde{\mu}_{j,N_j} = [\,\mu_{j,1} \quad \mu_{j,2} \quad \ldots \quad \mu_{j,N_j}\,]'$ and $\tilde{\mu}^*_{j,T} = [\,\mu^*_{j,1} \quad \mu^*_{j,2} \quad \ldots \quad \mu^*_{j,T}\,]'$, $j = 0, 1$, we note that the priors for $\tilde{\mu}^*_{0,T}$ and $\tilde{\mu}^*_{1,T}$ are derived from the priors for $\tilde{\mu}_{0,N_0}$ and $\tilde{\mu}_{1,N_1}$ along with their implied counterfactual priors $\tilde{\mu}_{0,N_1} = [\,\mu_{0,1} \quad \ldots \quad \mu_{0,N_1}\,]'$ and $\tilde{\mu}_{1,N_0} = [\,\mu_{1,1} \quad \ldots \quad \mu_{1,N_0}\,]'$. By additionally defining $\tilde{S}_T = [\,S_1 \quad S_2 \quad \ldots \quad S_T\,]'$, the full specification for the priors can be summarized as:

<u>*Summary of the Prior*</u>

$$
\begin{aligned}
&p(\tilde{\mu}_{0,N_0}, \tilde{\mu}_{1,N_1}, \tilde{\mu}_{0,N_1}, \tilde{\mu}_{1,N_0}, \tilde{S}_T, \mu_{0,0}, \mu_{1,0}, S_0, \delta, \sigma_e^2, \sigma_{\omega,0}^2, \sigma_{\omega,1}^2, \theta_0, \theta_1, p, q) \\
&\quad = p(\tilde{\mu}_{0,T}^*, \tilde{\mu}_{1,T}^*, \tilde{S}_T, \mu_{0,0}^*, \mu_{1,0}^*, S_0, \delta, \sigma_e^2, \sigma_{\omega,0}^2, \sigma_{\omega,1}^2, \theta_0, \theta_1, p, q) \\
&\quad = p(\tilde{\mu}_{0,T}^*, \tilde{\mu}_{1,T}^* | \mu_{0,0}^*, \mu_{1,0}^*, \tilde{S}_T, S_0, \delta, \sigma_e^2, \sigma_{\omega,0}^2, \sigma_{\omega,1}^2, \theta_0, \theta_1) \times p(\tilde{S}_T | S_0, p, q) \\
&\qquad \times p(\mu_{0,0}^*, \mu_{1,0}^*, S_0, \delta, \sigma_e^2, \sigma_{\omega,0}^2, \sigma_{\omega,1}^2, \theta_0, \theta_1, p, q) \\
&\quad = \left[\prod_{t=1}^{T} p(\mu_{0,t}^*, \mu_{1,t}^* | \mu_{0,t-1}^*, \mu_{1,t-1}^*, S_t, S_{t-1}, \sigma_{\omega,0}^2, \sigma_{\omega,1}^2, \theta_0, \theta_1) \right] \\
&\qquad \times \left[\prod_{t=1}^{T} p(S_t | S_{t-1}, p, q) \right] \times p(\mu_{0,0}^*, \mu_{1,0}^*) \times p(S_0) \times p(\delta | \sigma_e^2) \times p(\sigma_e^2) \\
&\qquad \times p(\theta_0 | \sigma_{\omega,0}^2) \times p(\sigma_{\omega,0}^2) \times p(\theta_1 | \sigma_{\omega,1}^2) \times p(\sigma_{\omega,1}^2) \times p(p, q),
\end{aligned}
\tag{41}
$$

where $p(\mu_{0,t}^*, \mu_{1,t}^* | \mu_{0,t-1}^*, \mu_{1,t-1}^*, S_t, S_{t-1}, \sigma_{\omega,0}^2, \sigma_{\omega,1}^2, \theta_0, \theta_1)$ is given by equations (33) and (34); $p(S_t | S_{t-1}, p, q)$ is given by the transition probabilities in equation (9); $p(\mu_{0,0}^*, \mu_{1,0}^*)$ is diffuse; $p(S_0)$ is given by the unconditional probabilities of S_t; $p(\delta | \sigma_e^2)$, $p(\theta_0 | \sigma_{\omega,0}^2)$ and $p(\theta_1 | \sigma_{\omega,1}^2)$ are independent normals; $p(\sigma_e^2)$, $p(\sigma_{\omega,0}^2)$, and $p(\sigma_{\omega,1}^2)$ are independent inverted Gamma's; $p(q, p)$ are independent Beta's.

Outline of the MCMC Procedure

Step 0:

Initialize the parameters of the model $\tilde{\psi} = [\,\delta \quad \sigma_e^2 \quad \theta_0 \quad \sigma_{\omega,0}^2 \quad \theta_1 \quad \sigma_{\omega,1}^2 \quad q \quad p\,]'$ and the states $\tilde{S}_T = [\,S_1 \quad S_2 \quad \dots \quad S_T\,]'$.

Step 1:

Generate $\tilde{\mu}_{0,T}^* = [\,\mu_{0,1}^* \quad \mu_{0,2}^* \quad \dots \quad \mu_{0,T}^*\,]'$ and $\tilde{\mu}_{1,T}^* = [\,\mu_{1,1}^* \quad \mu_{1,2}^* \quad \dots \quad \mu_{1,T}^*\,]'$ conditional on $\tilde{\psi}$, $\tilde{S}_T$, and data $\tilde{Y}_T = [\,y_1 \quad y_2 \quad \dots \quad y_T\,]'$. This step is based on the state-space representation of the model in equations (39) and (40).

Step 2:

Generate $\tilde{S}_T$ conditional on $\tilde{\mu}_{0,T}^*$ and $\tilde{\mu}_{1,T}^*$; parameters $\tilde{\psi}$; and data $\tilde{Y}_T$. This step is based on equation (39) and the transition probabilities in equation (9).

Step 3:

Generate θ_0, θ_1, $\sigma_{\omega,0}^2$ and $\sigma_{\omega_1}^2$, conditional on $\tilde{\mu}_{0,T}^*$, $\tilde{\mu}_{1,T}^*$, and $\tilde{S}_T$. This step is based on equations (26)-(29), by recovering $\tilde{\mu}_{0,N_0}$, $\tilde{\mu}_{1,N_1}$, $\tilde{\mu}_{0,N_1}$ and $\tilde{\mu}_{1,N_0}$ from $\tilde{\mu}_{0,T}^*$ and $\tilde{\mu}_{1,T}^*$, as

implied by the equivalence of equations (30)-(31) and equations (33)-(34).

Step 4:

Generate δ and σ_e^2, conditional on $\tilde{\mu}_{0,T}^*$, $\tilde{\mu}_{1,T}^*$,, $\tilde{S}_T$ and $\tilde{Y}_T$. This step is based on equation (39).

Step 5: Generate q and p conditional on $\tilde{S}_T$.

4.2. Details of the MCMC Procedure

4.2.1. Generating $\tilde{\mu}_{0,T}^*$ and $\tilde{\mu}_{1,T}^*$ conditional on $\tilde{S}_T$, parameters $\tilde{\psi}$, and data $\tilde{Y}_T$.

Conditional on $\tilde{S}_T$, equations (39)-(40) form a linear state-space model for the extended model in Section 3.2. This allows us to employ a slightly modified version of the procedure proposed by Carter and Kohn (1994). The conditional joint posterior distribution of $\tilde{\mu}_{0,T}^*$ and $\tilde{\mu}_{1,T}^*$ can be decomposed as:

$$p(\tilde{\mu}_{0,T}^*, \tilde{\mu}_{1,T}^* | \tilde{Y}_T, \tilde{S}_T, \tilde{\Psi}) = p(\mu_{0,T}^*, \mu_{1,T}^* | \tilde{Y}_T, \tilde{S}_T, \tilde{\psi}) \prod_{t=1}^{T-1} p(\mu_{0,t}^*, \mu_{1,t}^* | \mu_{0,t+1}^*, \mu_{1,t+1}^*, \tilde{Y}_t, \tilde{S}_T, \tilde{\psi}), \quad (42)$$

which suggests that we can sequentially generate $\mu_{0,t}^*$ and $\mu_{1,t}^*$ for $t = T, T-1, \ldots, 2, 1$. Note that, for identification of the model, we need to impose the restrictions, $\mu_{0,t}^* > 0$ and $\mu_{1,t}^* < 0$ for all t.

We run the Kalman filter for the state-space model given by equations (39)-(40) in order to obtain and save $\mu_{t|t}^* = E(\mu_t^* | \tilde{Y}_t, \tilde{S}_t, \tilde{\psi})$ and $P_{t|t} = Cov(\mu_t^* | \tilde{Y}_t, \tilde{S}_t, \tilde{\psi})$ for $t = 1, 2, ..., T$, where $\tilde{Y}_t = [\, y_1 \quad y_2 \quad \ldots \quad y_t \,]'$.

For $t = T$, we generate $\mu_T^* = [\, \mu_{0,T}^* \quad \mu_{1,T}^* \,]'$ from the joint normal distribution

$$\mu_T^* \; | \tilde{Y}_T, \tilde{S}_T, \tilde{\psi} \;\sim\; N(\mu_{T|T}^*, P_{T|T}). \quad (43)$$

For $t = T-1, T-2, .., 1$, we generate $\mu_t^* = [\, \mu_{0,t}^* \quad \mu_{1,t}^* \,]'$ conditional on $\mu_{t+1}^* = [\, \mu_{0,t+1}^* \quad \mu_{1,t+1}^* \,]'$. For this purpose, we first calculate

$$\mu_{t|t,\mu_{t+1}^*}^* = E(\mu_t^* | \tilde{Y}_t, \mu_{t+1}^*, \tilde{S}_T, \tilde{\psi}) = \mu_{t|t}^* + P_{t|t} F_{t+1}' (F_{t+1} P_{t|t} F_{t+1}' + \Omega_{t+1})^{-1} (\mu_{t+1}^* - F_{t+1} \mu_{t|t}^*) \quad (44)$$

and

$$P_{t|t,\mu_{t+1}} = Cov(\mu_t^*|\tilde{Y}_t, \mu_{t+1}^*, \tilde{S}_T, \tilde{\psi}) = P_{t|t} - P_{t|t}F_{t+1}^{'}(F_{t+1}P_{t|t}F_{t+1}^{'} + \Omega_{t+1})^{-1}F_{t+1}P_{t|t}. \quad (45)$$

Then, we can generate $\mu_{0,t}^*$ and $\mu_{1,t}^*$ in the following way:

i) If $S_t = 0$ and $S_{t+1} = 1$, we set $\mu_{0,t}^* =$ (1,1) element of $\mu_{t|t,\mu_{t+1}^*}^*$, and generate $\mu_{1,t}^*$ from the following distribution:

$$\mu_{1,t}^*|\mu_{t+1}^*, \tilde{Y}_t, \tilde{S}_T, \tilde{\psi} \ \sim N(\mu_{t|t,\mu_{t+1}^*}^*(2,1), \ P_{t|t,\mu_{t+1}^*}(2,2)), \quad (46)$$

where $\mu_{t|t,\mu_{t+1}^*}^*(2,1)$ and $P_{t|t,\mu_{t+1}^*}(2,2)$ are the (2,1) element of $\mu_{t|t,\mu_{t+1}^*}^*$ and the (2,2) element of $P_{t|t,\mu_{t+1}^*}$, respectively.

ii) If $S_t = 1$ and $S_{t+1} = 0$, we set $\mu_{1,t}^* =$ (2,1) element of $\mu_{t|t,\mu_{t+1}^*}^*$, and generate $\mu_{0,t}^*$ from the following distribution:

$$\mu_{0,t}^*|\mu_{t+1}^*, \tilde{Y}_t, \tilde{S}_T, \tilde{\psi} \ \sim N(\mu_{t|t,\mu_{t+1}^*}^*(1,1), \ P_{t|t,\mu_{t+1}^*}(1,1)), \quad (47)$$

where $\mu_{t|t,\mu_{t+1}^*}^*(1,1)$ and $P_{t|t,\mu_{t+1}^*}(1,1)$ are the (1,1) element of $\mu_{t|t,\mu_{t+1}^*}^*$ and the (1,1) element of $P_{t|t,\mu_{t+1}^*}$, respectively.

iii) Otherwise, we set $\mu_{0,t}^* =$ (1,1) element of $\mu_{t|t,\mu_{t+1}^*}^*$ and $\mu_{1,t}^* =$ (2,1) element of $\mu_{t|t,\mu_{t+1}^*}^*$.

4.2.2. Generating $\tilde{S}_T$ conditional on $\tilde{\mu}_{0,T}^*$, $\tilde{\mu}_{1,T}^*$, parameters $\tilde{\psi}$, and data $\tilde{Y}_T$

We employ a modified version of Albert and Chib's (1993) single-move Gibbs sampling for generating S_t, $t = 1, 2, ..., T$, conditional on $\tilde{S}_{\neq t} = [\,S_1 \quad \ldots S_{t-1} \quad S_{t+1} \quad \ldots \quad S_T\,]'$ and other variates. The key is in appropriately evaluating the predictive densities of y_t under two possible alternative regimes at time t (i.e., for $S_t = 0$ and for $S_t = 1$). However, unlike in the Hamilton model (1989) with constant mean growth rates (μ_0 and μ_1), the mean growth rates during recessions or booms in our model are not always defined, as discussed in the earlier sections. For example, conditional on $S_t = 1$ in the $(j-1)-th$ iteration of the MCMC procedure, only μ_{1,τ_1} is defined and μ_{0,τ_1} is not. The difficulty is that, when evaluating the predictive densities of y_t under two alternative regimes at the $j-th$ iteration of the MCMC procedure, we need μ_{0,τ_1} as well as μ_{1,τ_1}. We overcome this difficulty by taking advantage of the counterfactual priors in (28)-(29) as derived from the hierarchical priors in (26)-(27).

Note that $\mu_{0,t}^*$ and $\mu_{1,t}^*$ in equations (33)-(34) summarize both the hierarchical priors and the counterfactual priors for the mean growth rates under two alternative regimes, for all t.

Thus, the method for generating $\tilde{S}_t$ conditional on $\tilde{S}_{\neq t}$ and other variates is the same as in Albert and Chib (1993), except that we use $\mu_{0,t}^*$ and $\mu_{1,t}^*$ as the mean growth rates under two possible alternative regimes at each point in time. As in Albert and Chib (1993), $p(S_t|\tilde{Y}_T, \tilde{S}_{\neq t}, \tilde{\mu}_{0,T}^*, \tilde{\mu}_{1,T}^*, \tilde{\psi})$ can be derived as:

$$p(S_t|\tilde{Y}_T, \tilde{S}_{\neq t}, \tilde{\mu}_{0,T}^*, \tilde{\mu}_{1,T}^*, \tilde{\psi}) \propto Pr(S_t|S_{t-1})Pr(S_{t+1}|S_t)p(y_t|\tilde{Y}_{t-1}, S_t, \mu_{0,t}^*, \mu_{1,t}^*, \tilde{\psi}), \tag{48}$$

where

$$p(y_t|\tilde{Y}_{t-1}, S_t, \mu_{0,t}^*, \mu_{1,t}^*, \tilde{\psi}) = \frac{1}{\sqrt{2\pi\sigma_e^2}} exp\left(-\frac{1}{2\sigma_e^2}(y_t - \delta - \mu_{S_t,t}^*)^2\right). \tag{49}$$

Then, S_t can be generated from

$$Pr[S_t = 1|\tilde{Y}_T, \tilde{S}_{\neq t}, \tilde{\mu}_{0,T}^*, \tilde{\mu}_{1,T}^*, \tilde{\psi})] = \frac{p(S_t = 1|\tilde{Y}_T, \tilde{S}_{\neq t}, \tilde{\mu}_{0,T}^*, \tilde{\mu}_{1,T}^*, \tilde{\psi})}{\sum_{j=0}^{1} p(S_t = j|\tilde{Y}_T, \tilde{S}_{\neq t}, \tilde{\mu}_{0,T}^*, \tilde{\mu}_{1,T}^*, \tilde{\psi})}. \tag{50}$$

Note that, in Albert and Chib's (1993) procedure for the Hamilton model, they have $\mu_{S_t,t}^* = \mu_{S_t}$, $S_t = 0, 1$.

4.2.3. Generating θ_0, θ_1, $\sigma_{\omega,0}^2$ and $\sigma_{\omega,1}^2$, conditional on $\tilde{\mu}_{0,T}^*$, $\tilde{\mu}_{1,T}^*$, and $\tilde{S}_T$

For given $\tilde{S}_T$, we first extract $\tilde{\mu}_{0,N_0} = [\,\mu_{0,1} \quad \dots \quad \mu_{0,N_0}\,]'$ and $\tilde{\mu}_{1,N_1} = [\,\mu_{1,1} \quad \dots \quad \mu_{1,N_1}\,]'$, $\tilde{\mu}_{0,N_1} = [\,\mu_{0,1} \quad \dots \quad \mu_{0,N_1}\,]'$ and $\tilde{\mu}_{1,N_0} = [\,\mu_{1,1} \quad \dots \quad \mu_{1,N_0}\,]'$ from $\tilde{\mu}_{0,T}^* = [\,\mu_{0,1}^* \quad \dots \quad \mu_{0,T}^*\,]'$ and $\tilde{\mu}_{1,T}^* = [\,\mu_{1,1}^* \quad \dots \quad \mu_{1,T}^*\,]'$, as implied by the equivalence of equations (30)-(31) and (33)-(34). For example, $\tilde{\mu}_{0,N_0}$ and $\tilde{\mu}_{1,N_0}$ are the collections of $\mu_{0,t}^*$'s and $\mu_{1,t}^*$'s for which $S_{t-1} = 1$ and $S_t = 0$ for $t = 2, 3, ..., T$; $\tilde{\mu}_{0,N_1}$ and $\tilde{\mu}_{1,N_1}$ are the collections of $\mu_{0,t}^*$'s and $\mu_{1,t}^*$'s for which $S_{t-1} = 0$ and $S_t = 1$ for $t = 2, 3, ..., T$.

Then, based on equations (26)-(27), θ_0 and θ_1 can be generated conditional on $\sigma_{\omega,0}^2$ and $\sigma_{\omega,1}^2$; and then $\sigma_{\omega,0}^2$ and $\sigma_{\omega,1}^2$ can be generated conditional on θ_0 and θ_1. The prior and posterior distributions for generating these parameters are described below.

<u>*Prior*</u>

$$\theta_j \sim N(\underline{\theta}_j, \underline{\Sigma}_{\theta_j}), \quad j = 0, 1 \tag{51}$$

$$\sigma^2_{\omega,j} \;\sim\; IG\left(\frac{\nu_{\omega,j}}{2}, \frac{h_{\omega,j}}{2}\right), \quad j = 0, 1, \tag{52}$$

<u>*Posterior*</u>

$$\theta_j \mid \tilde{\mu}^*_{0,T}, \tilde{\mu}^*_{1,T}, \tilde{S}_T, \sigma^2_{\omega,0}, \sigma^2_{\omega,1} \sim N(\bar{\theta}_j, \bar{\Sigma}_{\theta_j}), \quad j = 0, 1, \tag{53}$$

$$\sigma^2_{\omega,j} \mid \theta_j, \tilde{\mu}^*_{0,T}, \tilde{\mu}^*_{1,T}, \tilde{S}_T \;\sim IG\left(\frac{\nu_{\omega,j} + N_j}{2}, \frac{h_{\omega,j} + \sum_{\tau_j=1}^{N_j} \omega^2_{j,\tau_j}}{2}\right), \quad j = 0, 1, \tag{54}$$

where

$$\bar{\theta}_j = \bar{\Sigma}_{\theta_j}\left(\underline{\Sigma}^{-1}_{\theta_j}\underline{\theta}_j + \frac{1}{\sigma^2_{\omega,j}}\sum_{\tau_j=1}^{N_j}(\pi_i\mu_{i,\tau_i'} + \pi_j\mu_{j,\tau_i'})(\mu_{j,\tau_j} - \mu_{j,\tau_i'})\right), \tag{55}$$
$$(j,i) = (0,1),\ (1,0)$$

$$\bar{\Sigma}_{\theta_j} = \left(\underline{\Sigma}^{-1}_{\theta_j} + \frac{1}{\sigma^2_{\omega,j}}\sum_{\tau_j=1}^{N_j}(\pi_i\mu_{i,\tau_i'} + \pi_j\mu_{j,\tau_i'})^2\right)^{-1}, \tag{56}$$

$$\omega_{j,\tau_j} = \mu_{j,\tau_j} - \mu_{j,\tau_i'} - \theta_j(\pi_i\mu_{i,\tau_i'} + \pi_j\mu_{j,\tau_i'}), \quad (j,i) = (0,1), (1,0), \tag{57}$$

and $\mu_{i,\tau_i'}$ is the mean growth rate during a regime right before the $\tau_j - th$ episode of boom ($j = 0$) or recession ($j = 1$).

4.2.4. Generating δ and σ^2_e, conditional on $\tilde{\mu}^*_{0,T}$, $\tilde{\mu}^*_{1,T}$, $\tilde{S}_T$, and $\tilde{Y}_T$

This step is based on equation (39). Conditional on $\tilde{S}_T$, $\tilde{\mu}^*_{0,T}$, $\tilde{\mu}^*_{1,T}$ and $\tilde{Y}_T$, we define $y^*_t = y_t - (1 - S_t)\mu^*_{0,t} - S_t\mu^*_{1,t}$, $t = 1, 2, ..., T$. Then, we have $y^*_t = \delta + e_t$. Based on this, the conditional posterior distributions for the δ and σ^2_e parameters can be easily derived. The prior and posterior distributions are given below:

<u>*Prior*</u>

$$\delta \sim N(\underline{\delta}, \underline{\Sigma}_\delta), \tag{58}$$

$$\sigma_e^2 \ \sim \ IG\left(\frac{\nu_e}{2}, \frac{h_e}{2}\right), \quad j = 0, 1, \tag{59}$$

<u>*Posterior*</u>

$$\delta \mid \tilde{\mu}^*_{0,T}, \tilde{\mu}^*_{1,T}, \tilde{S}_T, \sigma_e^2, \tilde{y}_T \ \sim N(\bar{\delta}, \bar{\Sigma}_\delta), \tag{60}$$

$$\sigma_e^2 \mid \delta, \tilde{\mu}^*_{0,T}, \tilde{\mu}^*_{1,T}, \tilde{S}_T, \tilde{Y}_T \ \sim IG\left(\frac{\nu_e + T}{2}, \frac{h_e + \sum_{t=1}^T (y_t^* - \delta)^2}{2}\right), \quad j = 0, 1, \tag{61}$$

where

$$\bar{\Sigma}_\delta = \left(\underline{\Sigma}_\delta^{-1} + \frac{T}{\sigma_e^2}\right)^{-1} \tag{62}$$

and

$$\bar{\delta} = \bar{\Sigma}_\delta \left(\underline{\Sigma}_\delta^{-1} \underline{\delta} + \frac{1}{\sigma_e^2} \sum_{t=1}^T y_t^*\right). \tag{63}$$

4.2.5. Generating q and p conditional on $\tilde{S}_T$

We employ the following Beta priors for q and p:

<u>*Prior*</u>

$$q \sim Beta(u_{00}, u_{01}), \tag{64}$$

$$p \sim Beta(u_{11}, u_{10}), \tag{65}$$

where u_{ij}, $i, j = 0, 1$, are the hyper-parameters. Then the posterior distribution can be derived as:

<u>*Posterior*</u>

$$p|\tilde{S}_T \sim Beta(u_{11} + n_{11}, u_{10} + n_{10}), \tag{66}$$

$$q|\tilde{S}_T \sim Beta(u_{00} + n_{00}, u_{01} + n_{01}), \tag{67}$$

where n_{ij} refers to the total number of transitions from state i to state j.

5. An Application to U.S. Real GDP Growth Data

We apply the proposed model and the MCMC procedure presented in Section 4 to postwar U.S. real GDP growth data that covers the sample period of 1947Q4 to 2011Q3. The results for the proposed model are compared to those of the Hamilton model (1989) with constant regime-specific parameters.

Our preliminary results suggest that serial correlation in the x_t term is important for the Hamilton model (1989) with constant regime-specific means. However, we find that no serial correlation in the x_t term is necessary for the proposed model with evolving regime-specific means. For both models, we incorporate a one-time structural break in σ_e^2 in equation (8), in order to account for the Great Moderation (McConnell and Perez-Quiros (2000) and Kim and Nelson (1999)). We also incorporate a one-time structural break in the long-run growth of real GDP (the δ parameter in equation (21)). What follows describes the Hamilton model and the proposed model with these features:

Constant Regime-Specific Mean Growth Rates

$$y_t = \delta_{D_{2,t}} + (1 - S_t)\mu_0 + S_t\mu_1 + x_t$$

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + e_t, \quad e_t|D_{1,t} \sim i.i.d.N(0, \sigma^2_{e,D_{1,t}})$$

$$\mu_0 > 0, \quad \mu_1 < 0$$

$$Pr[S_t = 0|S_{t-1} = 0] = q, \quad Pr[S_t = 1|S_{t-1} = 1] = p$$

$$Pr[D_{i,t} = 0|D_{i,t-1} = 0] = q_i, \quad Pr[D_{i,t} = 1|D_{i,t-1} = 1] = 1 \quad i = 1, 2.$$

Evolving Regime-Specific Mean Growth Rates

$$y_t = \delta_{D_{2,t}} + (1 - S_t)\mu_{0,\tau} + S_t\mu_{1,\tau} + e_t, \quad e_t|D_{1t} \sim i.i.d.N(0, \sigma^2_{e,D_{1t}})$$

$$\mu_{0,\tau} = \mu_{0,\tau-1} + \theta_0(\pi_0\mu_{0,\tau-1} + \pi_1\mu_{1,\tau-1}) + \omega_{0,\tau}, \quad \omega_{0,\tau} \sim i.i.d.N(0, (1 - S_t)\sigma^2_{\omega,0})$$

$$\mu_{1,\tau} = \mu_{1,\tau-1} + \theta_1(\pi_0\mu_{0,\tau-1} + \pi_1\mu_{1,\tau-1}) + \omega_{1,\tau}, \quad \omega_{1,\tau} \sim i.i.d.N(0, S_t\sigma^2_{\omega,1})$$

$$Pr[S_t = 0|S_{t-1} = 0] = q, \quad Pr[S_t = 1|S_{t-1} = 1] = p$$

$$Pr[D_{i,t} = 0|D_{i,t-1} = 0] = q_i, \quad Pr[D_{i,t} = 1|D_{i,t-1} = 1] = 1, \quad i = 1, 2,$$

$$\mu_{0,\tau} > 0, \quad \mu_{1,\tau} < 0, \quad for\ all\ \tau,\ j = 0, 1$$

$$-1 < 1 + \theta_0\pi_0 + \theta_1\pi_1 < 1,$$

where π_0 and π_1 are the unconditional probabilities. Conditional on S_t, if we rewrite the the first three equations of above model in the form of the standard unobserved-components model, we have:

$$y_t = \delta_{D_{2,t}} + (1 - S_t)\mu^*_{0,t} + S_t\mu^*_{1,t} + e_t, \quad e_t|D_{1t} \sim i.i.d.N(0, \sigma^2_{e,D_{1t}}),$$

$$\mu^*_{0,t} = \mu^*_{0,t-1} + \theta_0(d_{10,t} + d_{01,t})(\pi_0\mu^*_{0,t-1} + \pi_1\mu^*_{1,t-1}) + \omega^*_{0,t}, \quad \omega^*_{0,t} \sim N(0, d_{10,t}\sigma^2_{\omega,0}),$$

$$\mu^*_{1,t} = \mu^*_{1,t-1} + \theta_1(d_{10,t} + d_{01,t})(\pi_0\mu^*_{0,t-1} + \pi_1\mu^*_{1,t-1}) + \omega^*_{1,t}, \quad \omega^*_{1,t} \sim N(0, d_{01,t}\sigma^2_{\omega,1}),$$

$$t = 1, 2, ..., T; \quad \tau = 1, 2, ..., N_0 + N_1,$$

where N_0 and N_1 are the total numbers of the episodes of boom and recession, respectively, conditional on the states; and

$$\mu^*_{0,t} > 0, \quad \mu^*_{1,t} < 0, \quad for\ all\ t,$$

$$d_{ij,t} = \begin{cases} 1, & \text{if } S_{t-1} = i\ and\ S_t = j,\ \ j \neq i; \\ 0, & \text{otherwise.} \end{cases}$$

All inferences are based on 50,000 Gibbs simulations after discarding 10,000 burn-ins. Table 2 presents the prior and posterior moments of the parameters for the Hamilton model with AR(2) dynamics for x_t. In Figure 5, the posterior probabilities of recession are depicted against the NBER recessions (shaded areas). They suggest that the model does a poor job in identifying recessions. In Figure 6, the posterior mean growth rates obtained from the model are depicted against real GDP growth and the episode-specific mean growth rates for the NBER recessions or booms. The two measures of mean growth rates are quite different,

indicating that the Hamilton model does a poor job in estimating the mean growth rates that vary over time.

Table 3 presents the prior and posterior moments of the parameters for the model proposed in Section 3.2. With regime-specific mean growth rates evolving over different episodes of booms or recessions, we have a much sharper inference on the recession probabilities, as depicted in Figure 9. The posterior probabilities of recession inferred from the proposed model are in close agreement with the NBER recessions. The posterior mean growth rates obtained from the model, as depicted in Figure 10, are also in close agreement with the episode-specific mean growth rates for the NBER recessions or booms. [2]

Figure 11 depicts the cumulative probability of structural break in the conditional variance ($\sigma^2_{e,D_{1t}}$) of real GDP from the proposed model. As reported in the literature, the process of the Great Moderation, i.e., the structural break in the conditional variance, is fairly abrupt and concentrated around the mid-1980's. However, the nature of the structural break in the equilibrium long-run output growth seems ($\delta_{D_{2t}}$) to be quite different from what has been reported in the literature. While the literature suggests an abrupt decline after the first Oil Shock of the mid-1970's, the plot of the cumulative probability of structural break in Figure 12 suggests that the decline occurred steadily over a thirty-year period between the mid-1950s and the mid-1980s. It is interesting to note that the decline in the long-run equilibrium output growth that started in the mid-1950's ended just when the Great Moderation began. [3]

Posterior moments for the θ_0 and θ_1 parameters in Table 3 provide us information about how successful the central bank may have been in its attempts to maintain the economy at a long-run equilibrium growth path. Even though their posterior means are both negative as predicted, their posterior standard deviations seem to be somewhat too high to give us any decisive evidence. However, if we compare the prior and posterior distributions for these parameters as depicted in Figures 13.A and 13.B, we can infer that there exists relatively

[2] The posterior means of the standardized residuals obtained from the model show little evidence of serial correlation. The same is true for the squared standardized residuals. These indicate that the proposed model with AR(0) dynamics for the x_t term passes the usual diagnostic checks.

[3] We get similar results from the Hamilton model, as depicted in Figures 7 and 8.

more sample evidence in favor of $\theta_0 < 0$ than that in favor of $\theta_1 < 0$. [4] Figure 14 plots the impulse-response functions for the regime-specific mean growth rates with respect to a one standard deviation (SD) shock. [5] Of particular interest would be the comparison of $\frac{\partial \Delta\mu_{1,\tau+j}}{\partial \omega_{0,\tau}}$ and $-\frac{\partial \Delta\mu_{0,\tau+j}}{\partial \omega_{1,\tau}}$ depicted in the two graphs in the lower panel of Figure 14. As for the responses of the mean growth rates for recessions to a one standard-deviation boom shock ($\frac{\partial \Delta\mu_{1,\tau+j}}{\partial \omega_{0,\tau}}$), the 68% posterior bands are so wide that we find little evidence that they are negative. However, as for the responses of the mean growth rates for booms to a one standard-deviation recession shock ($-\frac{\partial \Delta\mu_{0,\tau+j}}{\partial \omega_{1,\tau}}$), we find some evidence that it is positive for $j = 1$. The results from the estimates of the θ_0 and θ_1 parameters or those from the impulse response analyses suggest that the central bank may have been relatively more effective in restoring the economy back to its long-run equilibrium growth path after unusually severe recessions than after unusually good booms. Thus, our empirical results provide partial, if not decisive, evidence that the central bank's long-run policy may have been asymmetric in response to unusually pronounced recessions and booms. [6]

6. Summary

As an economy evolves over time along with evolving institutions and policies, so do the dynamics of the business cycle. Over time, we thus may need bigger and more sophisticated empirical models which are capable of capturing the changes in the dynamics of the business cycle. The Great Moderation, i.e., the stabilization of the economy since the mid-1980s, is an example of such change. However, what is sometimes overlooked in empirical models of the business cycle is that the postwar booms and recessions are not all alike. For example, a two-state Markov-switching model of the business cycle, as proposed by Hamilton (1989), assumes that mean growth rates during all episodes of booms or recessions are the same.

[4] The results are robust with respect to alternative priors employed for the θ_0 and θ_1 parameters.

[5] Note that this shock causes the mean growth rate during the current episode of boom or recession to be different from that during the previous episode. The impulse response-functions are calculated based on equations (36)-(37).

[6] We assume that these unusually pronounced recessions or booms cause the economy to deviate from their long-run equilibrium growth path.

While this assumption may be valid for particular sample periods, it may not be a realistic one for a sample that covers the entire postwar period. This is why the Hamilton model fails to provide sharp inferences on two distinctive business cycle regimes when the sample period is extended beyond that employed by Hamilton (1989).

In this paper, within a two-state Markov-switching model, we assume that the mean growth rate for recession or boom undergoes a structural break whenever we face a new episode of recession or boom. We first consider the case in which each regime-specific mean growth rate evolves according to a random walk process over different episodes of boom or recession. We then derive and impose a condition for the existence of an equilibrium long-run growth rate for real output. As a consequence of this condition, we incorporate vector error correction dynamics for the two regime-specific mean growth rates.

When applied to the postwar (1947Q4-2011Q3) real GDP growth data, the proposed model considerably outperforms the Hamilton model (1989) with constant regime-specific mean growth rates, both in identifying recessions and in making inferences about the mean growth rates. The evolving nature of each regime-specific mean growth rate for booms or recessions is not the only feature of the U.S. postwar business cycle that we uncover in this paper. Another interesting finding is that the decline in the long-run equilibrium output growth was not abrupt. It started in the mid-1950's ended in the mid-1980's, coinciding with the beginning of the Great Moderation. This is in sharp contrast to the literature, which suggests an abrupt decline in the long-run output growth around the mid-1970s.

Furthermore, empirical results obtained from the proposed model provide partial, if not decisive, evidence that the central bank's long-run policy may have been asymmetric in response to unusually pronounced recessions and booms. The central bank has been relatively more effective in restoring the economy back to its long-run equilibrium growth path after unusually severe recessions than after unusually high booms.

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Table 1. Episode-Specific Mean Growth Rates of Real GDP During NBER Booms and Recessions [1947:IV - 2011:III]

Boom		Recession	
47:Q4 $\sim$ 48:Q3	1.37	48:Q4 $\sim$ 49:Q4	-0.28
50:Q1 $\sim$ 53:Q2	1.83	53:Q3 $\sim$ 54:Q2	-0.64
54:Q3 $\sim$ 57:Q2	0.98	57:Q3 $\sim$ 58:Q2	-0.55
58:Q3 $\sim$ 60:Q1	1.67	60:Q2 $\sim$ 61:Q1	-0.25
61:Q2 $\sim$ 69:Q3	1.24	69:Q4 $\sim$ 70:Q4	-0.12
71:Q1 $\sim$ 73:Q3	1.30	73:Q4 $\sim$ 75:Q1	-0.38
75:Q2 $\sim$ 79:Q4	1.09	80:Q1 $\sim$ 80:Q3	-0.64
80:Q4 $\sim$ 81:Q2	1.04	81:Q3 $\sim$ 82:Q4	-0.24
83:Q1 $\sim$ 90:Q2	1.06	90:Q3 $\sim$ 91:Q1	-0.45
91:Q2 $\sim$ 00:Q4	0.91	01:Q1 $\sim$ 01:Q4	0.02
02:Q1 $\sim$ 07:Q3	0.66	07:Q4 $\sim$ 09:Q2	-0.69
09:Q3 $\sim$ 11:Q3	0.59		
Mean	1.15		-0.38
Maximum	1.83		0.02
Minimum	0.59		-0.69
Standard Deviation	0.37		0.23

Table 2. Prior and Posterior Distributions: Hamilton model [Real GDP Growth: 1948:I - 2011:I]

$$y_t = \delta_{D_{2,t}} + (1 - S_t)\mu_0 + S_t\mu_1 + x_t,$$
$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + e_t, \quad e_t|D_{1,t} \sim i.i.d.N(0, \sigma^2_{e,D_{1,t}}),$$
$$\mu_0 > 0, \quad \mu_1 < 0$$
$$Pr[S_t = 0|S_{t-1} = 0] = q, \quad Pr[S_t = 1|S_{t-1} = 1] = p,$$
$$Pr[D_{i,t} = 0|D_{i,t-1} = 0] = q_i, \quad Pr[D_{i,t} = 1|D_{i,t-1} = 1] = 1, \quad i = 1, 2.$$

	Prior		Posterior		
	Mean	SD	Mean	SD	90% Bands
μ_0	0.500	1.000	0.097	0.086	$[0.006, 0.269]$
μ_1	-1.000	1.000	-1.424	0.408	$[-2.036, -0.658]$
ϕ_1	0.000	1.000	0.273	0.076	$[0.151, 0.400]$
ϕ_2	0.000	1.000	0.198	0.074	$[0.075, 0.319]$
σ_0^2	0.750	0.750	1.161	0.170	$[0.902, 1.461]$
σ_1^2	0.750	0.750	0.254	0.040	$[0.197, 0.327]$
q	0.900	0.090	0.963	0.021	$[0.925, 0.988]$
p	0.800	0.121	0.729	0.105	$[0.542, 0.894]$
q_1	0.986	0.014	0.991	0.006	$[0.978, 0.998]$
q_2	0.986	0.014	0.987	0.010	$[0.967, 0.998]$
δ_0	1.300	0.200	1.203	0.170	$[0.934, 1.490]$
δ_1	1.000	0.200	0.716	0.111	$[0.519, 0.891]$

Table 3. Prior and Posterior Distributions: Markov-Switching Model with Evolving Regime-Specific Mean Growth Rates [Real GDP Growth: 1948:I - 2011:I]

$$y_t = \delta_{D_{2,t}} + (1 - S_t)\mu_{0,\tau} + S_t\mu_{1,\tau} + e_t, \quad e_t|D_{1t} \sim i.i.d.N(0, \sigma^2_{e,D_{1t}})$$

$$\mu_{0,\tau} = \mu_{0,\tau-1} + \theta_0(\pi_0\mu_{0,\tau-1} + \pi_1\mu_{1,\tau-1}) + \omega_{0,\tau}, \quad \omega_{0,\tau} \sim i.i.d.N(0, (1 - S_t)\sigma^2_{\omega,0})$$

$$\mu_{1,\tau} = \mu_{1,\tau-1} + \theta_1(\pi_0\mu_{0,\tau-1} + \pi_1\mu_{1,\tau-1}) + \omega_{1,\tau}, \quad \omega_{1,\tau} \sim i.i.d.N(0, S_t\sigma^2_{\omega,1})$$

$$Pr[S_t = 0|S_{t-1} = 0] = q, \quad Pr[S_t = 1|S_{t-1} = 1] = p$$

$$Pr[D_{i,t} = 0|D_{i,t-1} = 0] = q_i, \quad Pr[D_{i,t} = 1|D_{i,t-1} = 1] = 1, \quad i = 1, 2,$$

$$\mu_{0,\tau} > 0, \quad \mu_{1,\tau} < 0, \quad for\ all\ \tau,$$

$$-1 < 1 + \theta_0\pi_0 + \theta_1\pi_1 < 1,$$

where π_0 and π_1 are the unconditional probabilities.

	Prior		Posterior		
	Mean	SD	Mean	SD	90% Bands
θ_0	-0.100	0.500	-0.351	0.297	$[-0.879, 0.081]$
θ_1	-0.100	0.500	-0.234	0.394	$[-0.882, 0.409]$
$\sigma^2_{\omega 0}$	0.333	0.236	0.110	0.039	$[0.063, 0.184]$
$\sigma^2_{\omega 1}$	0.333	0.236	0.148	0.061	$[0.077, 0.262]$
σ^2_0	0.750	0.750	0.810	0.134	$[0.611, 1.048]$
σ^2_1	0.750	0.750	0.232	0.042	$[0.174, 0.308]$
q	0.900	0.090	0.912	0.035	$[0.849, 0.958]$
p	0.800	0.121	0.787	0.065	$[0.667, 0.881]$
q_1	0.986	0.014	0.991	0.007	$[0.978, 0.998]$
q_2	0.986	0.014	0.987	0.010	$[0.967, 0.998]$
δ_0	1.300	0.200	1.245	0.138	$[1.030, 1.485]$
δ_1	1.000	0.200	0.613	0.110	$[0.427, 0.790]$

Figure 1. Real GDP Growth and Its Episode-Specific Means During NBER Booms and Recessions [1947:IV - 2011:III]

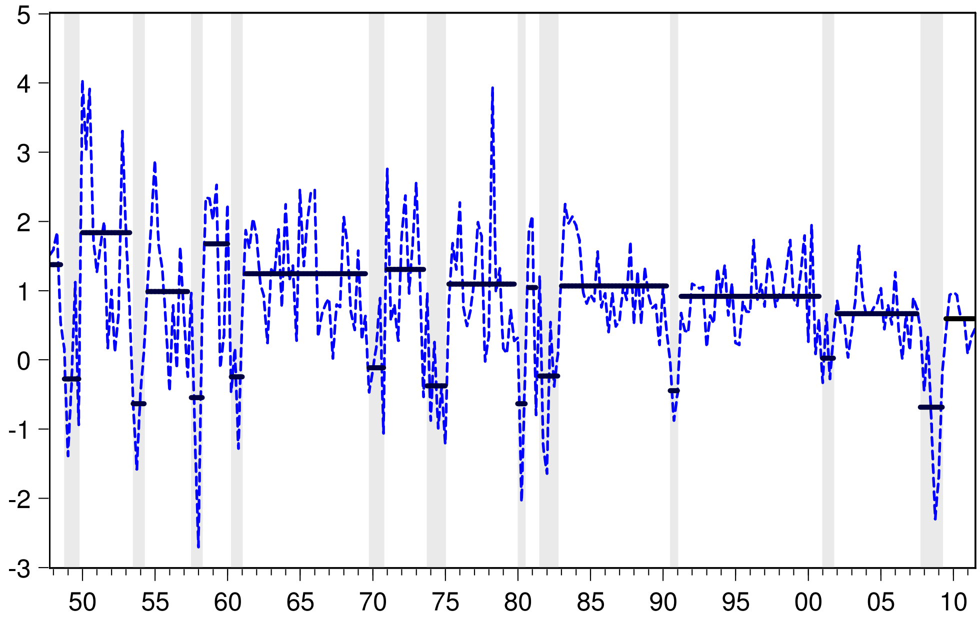
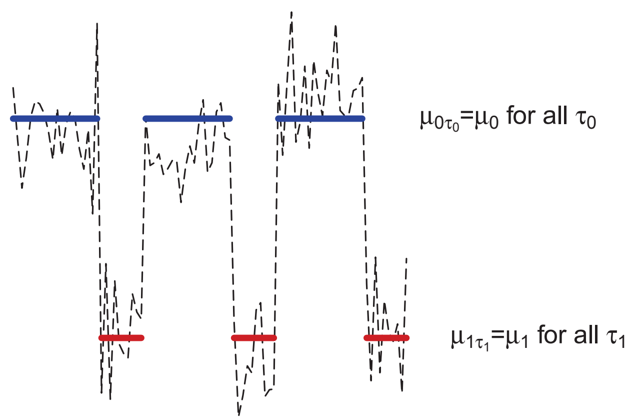
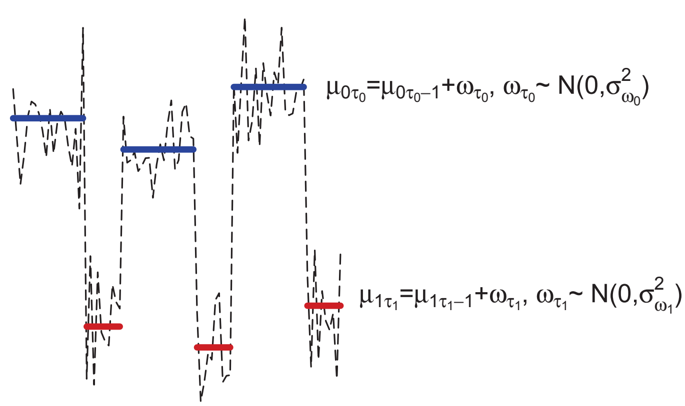


Figure 2. Comparison of Hamilton (1989) Model and the Proposed Model

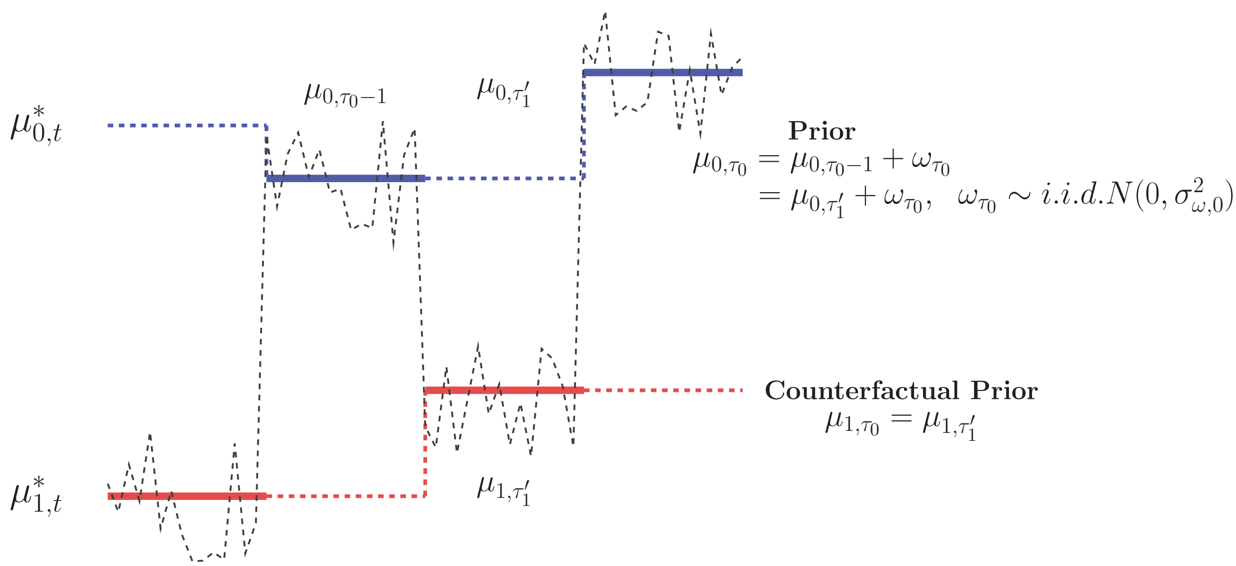


A. <u>Markov-Switching Model with Constant Regime-Specific Mean Growth Rates</u>
(Hamilton Model)

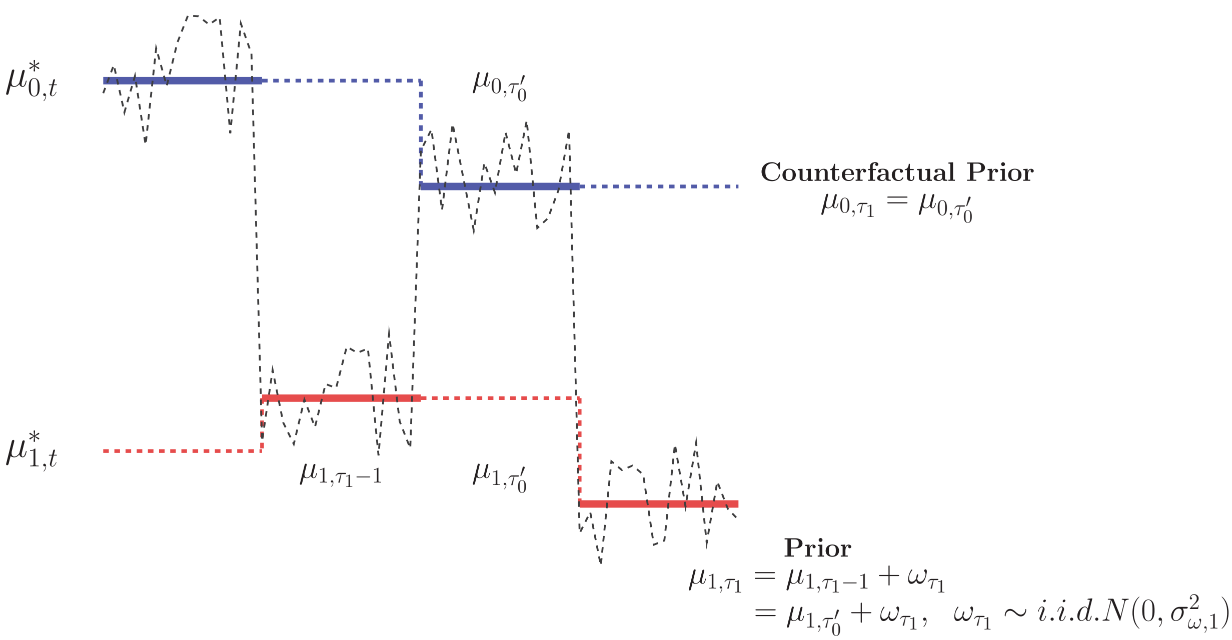


B. <u>Markov-Switching Model with Evolving Regime-Specific Mean Growth Rates</u>
(Proposed Model)

Figure 3. Priors and Counterfactual Priors: Random Walk for Regime-Specific Mean Growth Rates

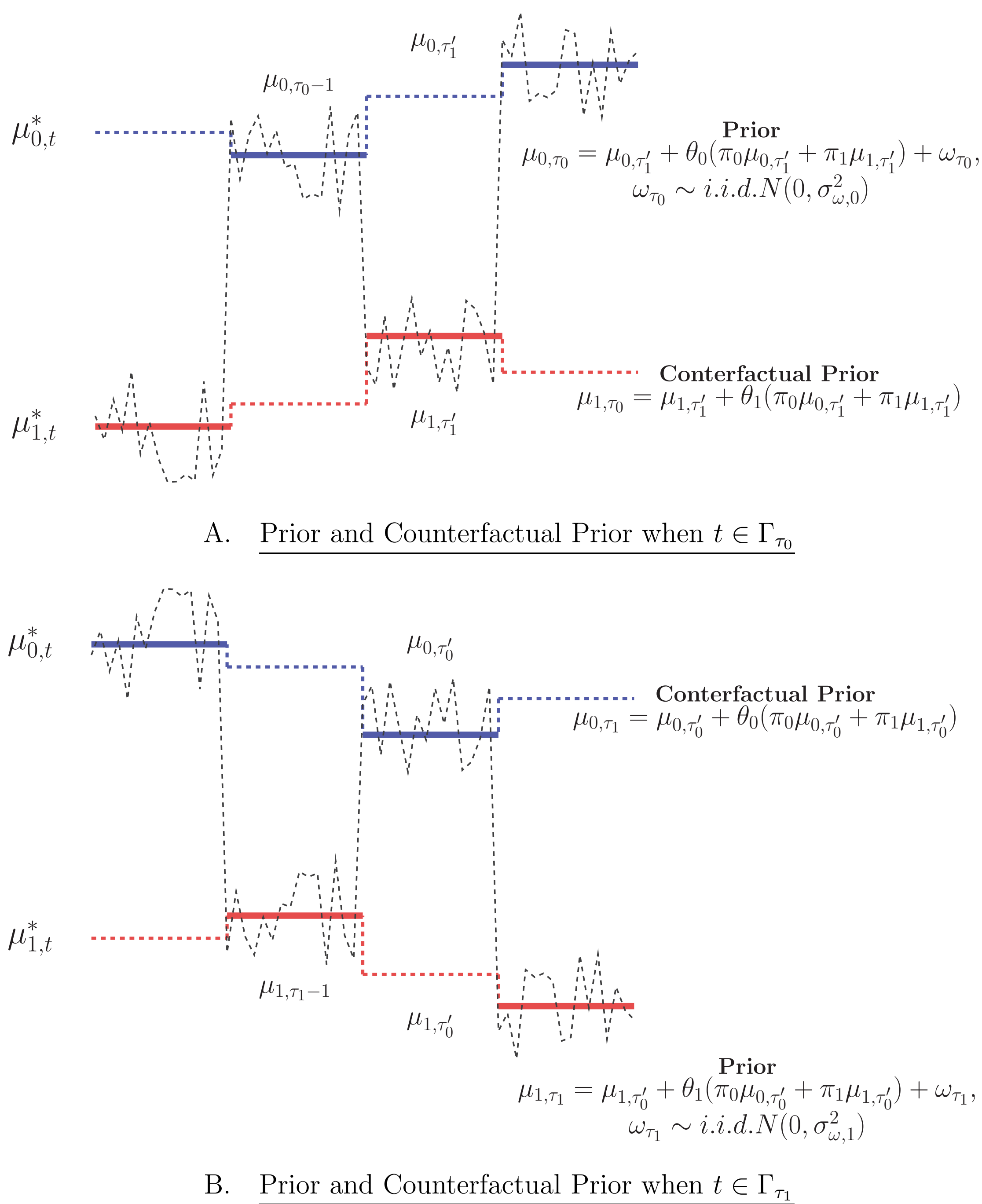


A. Prior and Counterfactual Prior when $t \in \Gamma_{\tau_0}$



B. Prior and Counterfactual Prior when $t \in \Gamma_{\tau_1}$

Figure 4. Priors and Counterfactual Priors: Vector Error Correction Dynamics for Regime-Specific Mean Growth Rates



A. Prior and Counterfactual Prior when $t \in \Gamma_{\tau_0}$

B. Prior and Counterfactual Prior when $t \in \Gamma_{\tau_1}$

Figure 5. Posterior Probability of Recession [Hamilton Model (1989)]

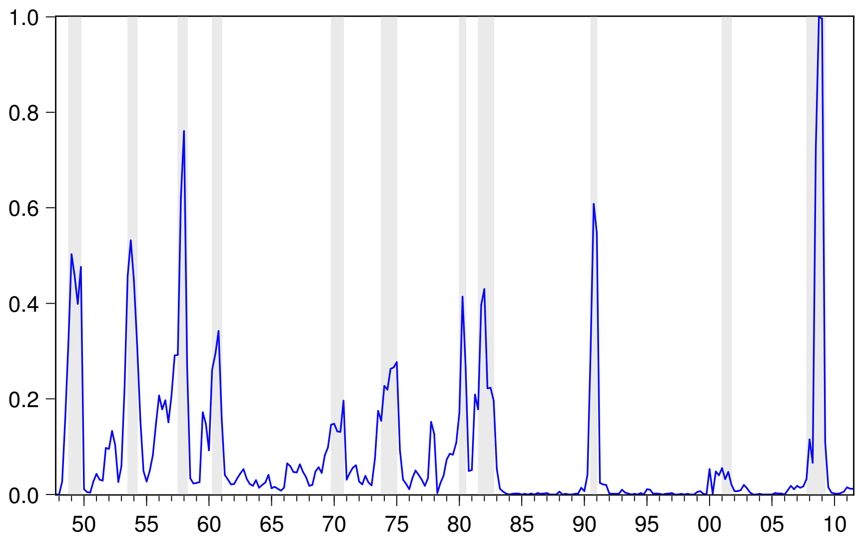


Figure 6. NBER Episode-Specific Mean Growth Rates and Posterior Mean Growths Rates: [Hamilton Model (1989)]

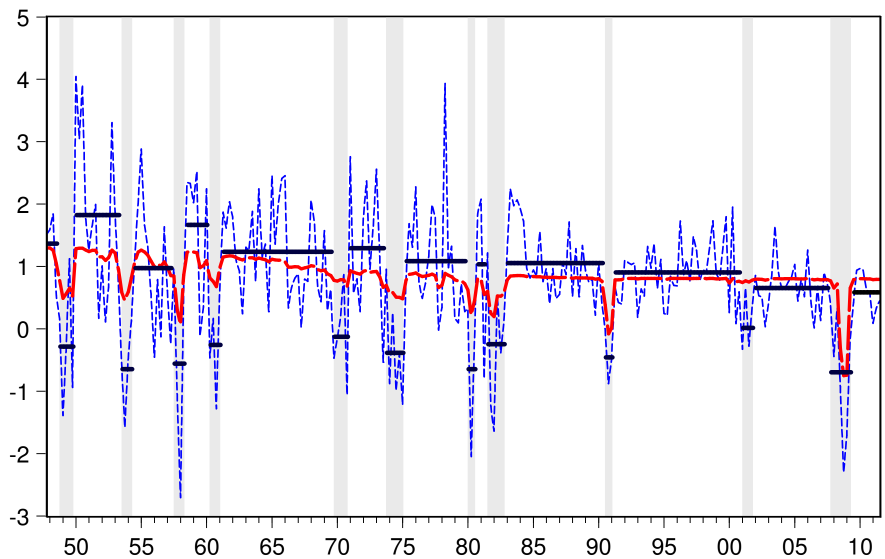


Figure 7. Posterior Cumulative Probability of Structural Break in Conditional Variance ($Pr[D_{1,t} = 1|\tilde{Y}_T$) [Hamilton Model (1989)]

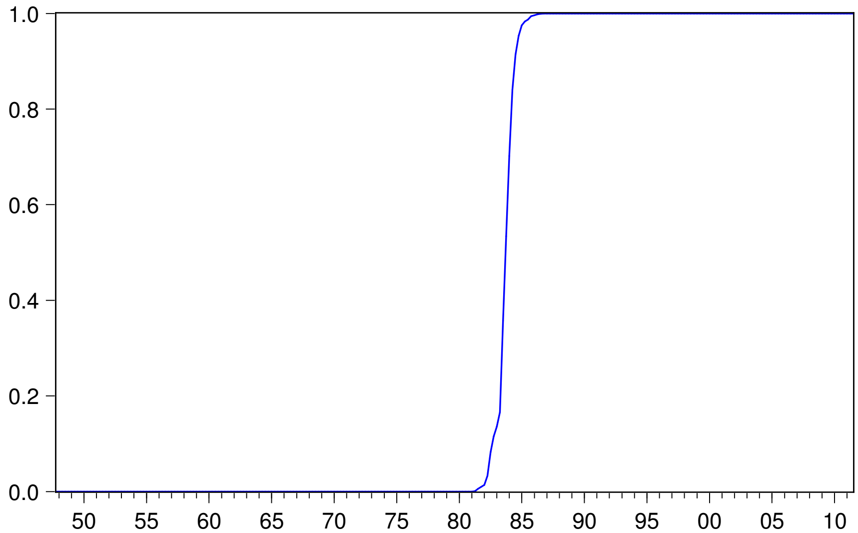


Figure 8. Posterior Cumulative Probability of Structural Break in Long-Run Growth Rate ($Pr[D_{2,t} = 1|\tilde{Y}_T$) [Hamilton Model (1989)]

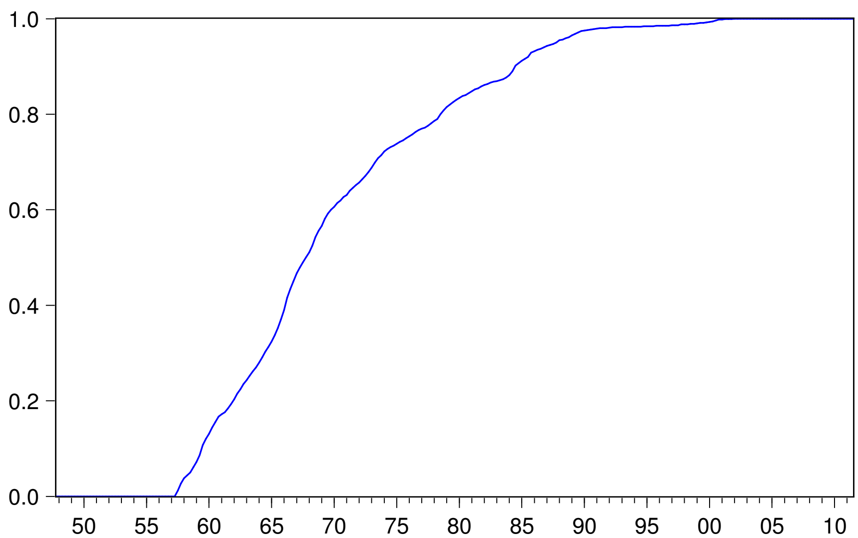


Figure 9. Posterior Probability of Recession [Proposed Model]

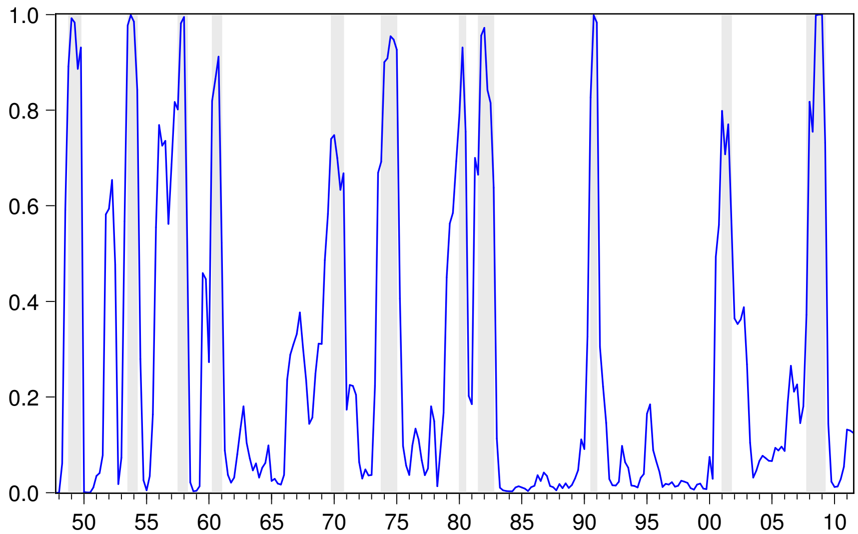


Figure 10. NBER Episode-Specific Mean Growth Rates and Posterior Mean Growths Rates: [Proposed Model]

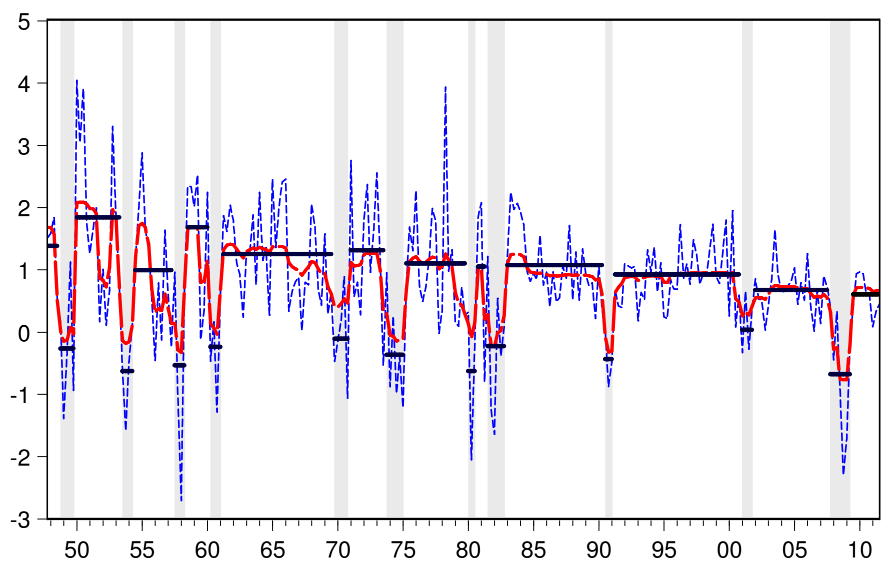


Figure 11. Posterior Cumulative Probability of Structural Break in Conditional Variance ($Pr[D_{1,t} = 1|\tilde{Y}_T)$ [Proposed Model]

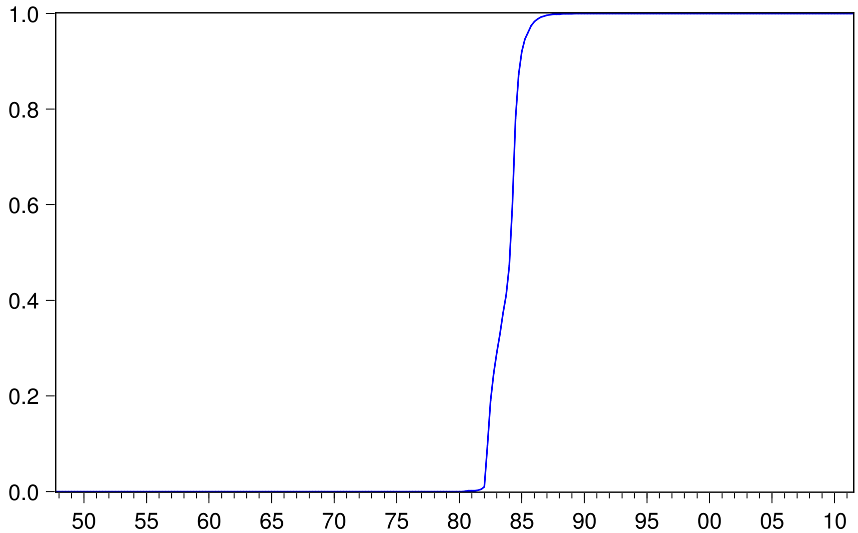


Figure 12. Posterior Cumulative Probability of Structural Break in Long-Run Growth Rate ($Pr[D_{2,t} = 1|\tilde{Y}_T)$ [Proposed Model]

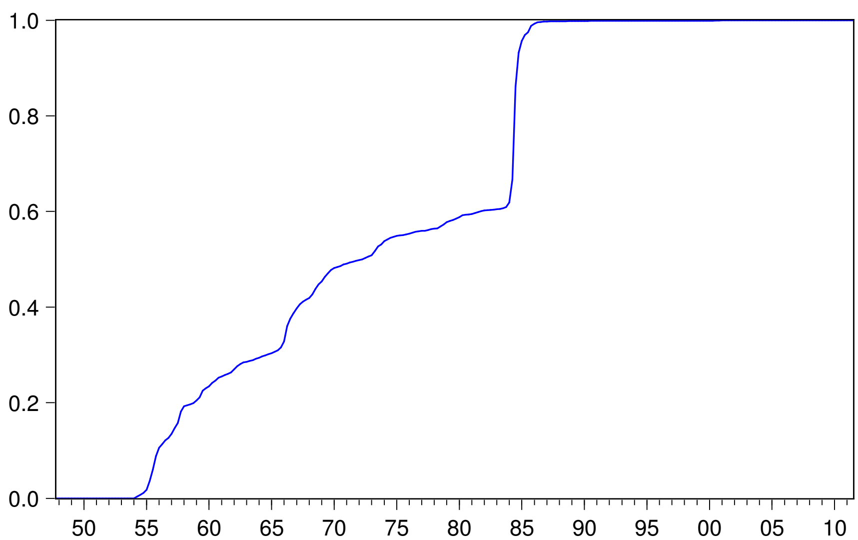
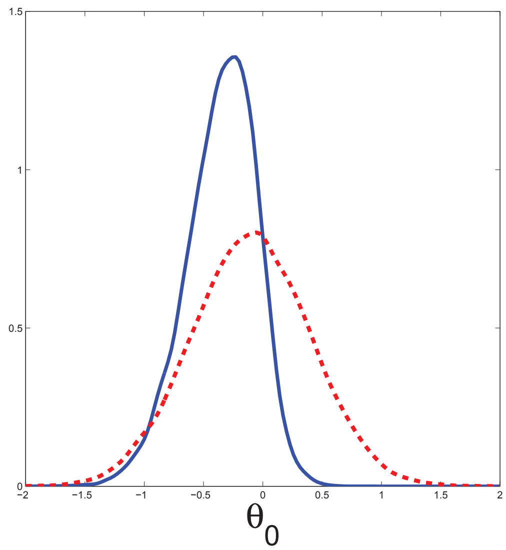
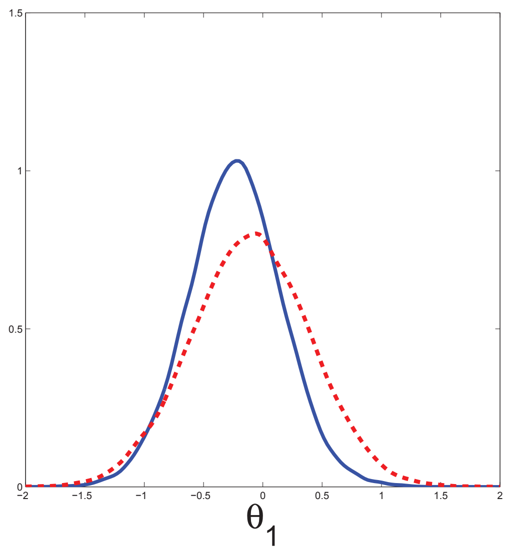


Figure 13. Prior and Posterior Distributions for Error Correction Coefficients: [Proposed Model]



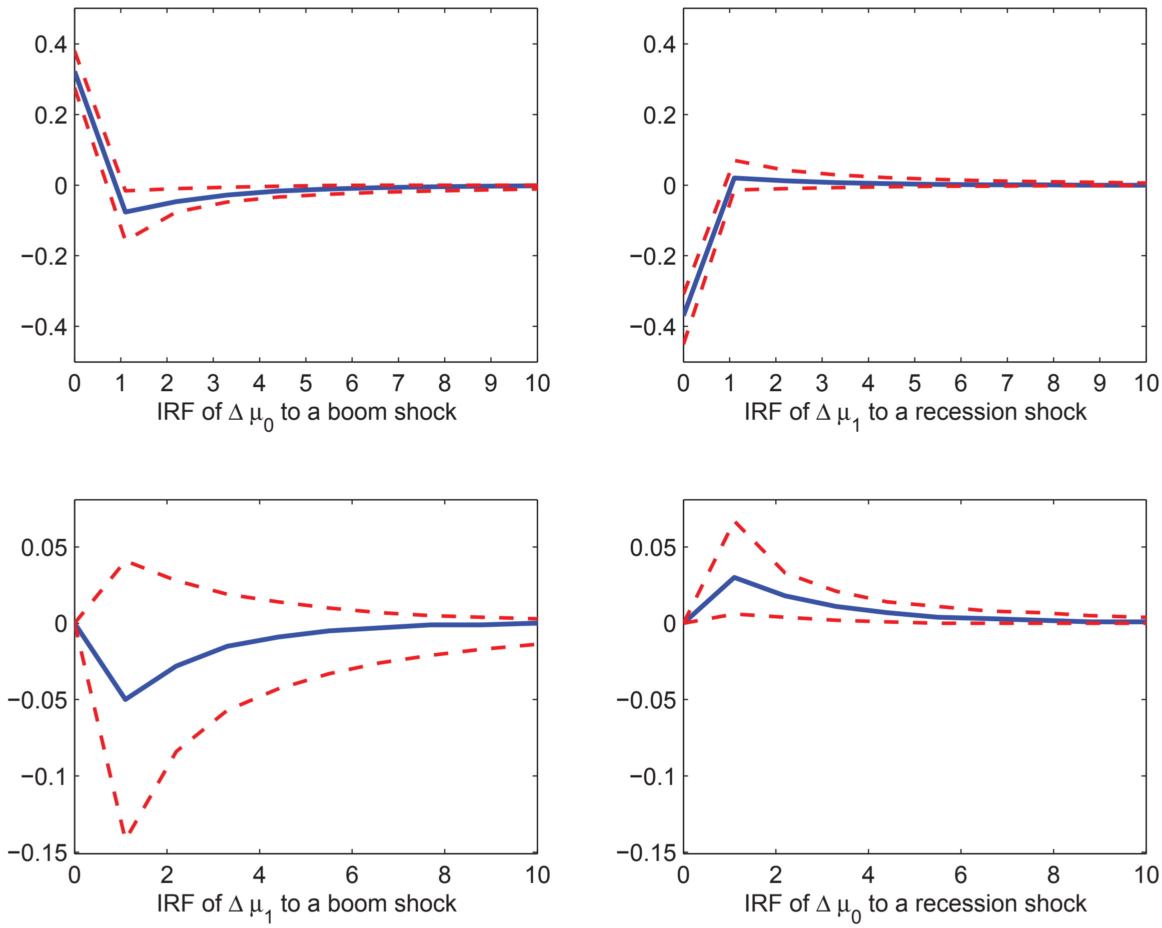
A. Error Correction Coefficient θ_0



B. Error Correction Coefficient θ_1

Note: The solid line and the dashed line represent the posterior distribution and the prior distribution, respectively.

Figure 14. Impulse Response Functions for the Regime-Specific Mean Growth Rates



Note: The solid line and the dashed line represent the posterior mean and the 68% posterior band, respectively.